Exercise 21

Solve the initial-value problem.

$$y'' - 6y' + 10y = 0, \quad y(0) = 2, \quad y'(0) = 3$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Plug these formulas into the ODE.

$$r^2e^{rx} - 6(re^{rx}) + 10(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 6r + 10 = 0$$

Solve for r.

$$r = \frac{6 \pm \sqrt{36 - 4(1)(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$$

$$r = \{3 - i, 3 + i\}$$

Two solutions to the ODE are $e^{(3-i)x}$ and $e^{(3+i)x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1e^{(3-i)x} + C_2e^{(3+i)x} \\ &= C_1e^{3x}e^{-ix} + C_2e^{3x}e^{ix} \\ &= e^{3x}(C_1e^{-ix} + C_2e^{ix}) \\ &= e^{3x}[C_1(\cos x - \sin x) + C_2(\cos x + i\sin x)] \\ &= e^{3x}[(C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x] \\ &= e^{3x}(C_3\cos x + C_4\sin x). \end{aligned}$$

Differentiate the general solution.

$$y'(x) = 3e^{3x}(C_3\cos x + C_4\sin x) + e^{3x}(-C_3\sin x + C_4\cos x)$$

Apply the initial conditions to determine C_3 and C_4.

$$y(0) = C_3 = 2$$

$$y'(0) = 3C_3 + C_4 = 3$$

Solving this system of equations yields $C_3 = 2$ and $C_4 = -3$. Therefore, the solution to the initial value problem is

$$y(x) = e^{3x}(2\cos x - 3\sin x).$$

Below is a graph of $y(x)$ versus x.